Formalizing the Semicircle Law in Lean

Haoyan Jiang, Richard Oh, Fred Rajasekaran, Kiran Sun, Paul Yoon

July 26, 2025

Chapter 1

Empirical Distribution

Lemma 1.0.1. *Let $(\mathcal{X}, d)$ be a Polish space. Let $\mu : \Omega \to \mathcal{P}(\mathcal{X})$ be measureable wiht respect to the cylinder σ-algebra on $\mathcal{P}(\mathcal{X})$. Let $f : \mathcal{X} \to \mathcal{Y}$ be measurable. define $\varphi : \Omega \to \mathcal{Y}$ by*

$$\varphi(\omega) = \int_{\mathcal{X}} f(x)\, d\mu(\omega)(x).$$

Then φ is a measurable map from $\Omega \to \mathcal{Y}$.

Chapter 2

Random Matrix Theory

Chapter 3

Convergence of Matrix Moments

3.1 Convergence in Expectation

Lemma 3.1.1 (Matrix Powers Entries)**.** *Let Y be an $n \times n$ matrix and $k \in \mathbb{N}$. Then, for each (i, j)-th entry of Y^k, we have:*

$$(Y^k)_{ij} = \sum_{1 \le i_2, \dots, i_k \le n} Y_{ii_2} Y_{i_2 i_3} \dots Y_{i_k j}$$

Proof. We proceed by induction on k.

Our base case is $k = 1$, then:

$$Y_n^1 = Y_n \quad \Rightarrow \quad [Y_n^1]_{ij} = Y_{ij},$$

which matches the formula since the summation over zero indices just gives the term Y_{ij}.

Our inductive step is to assume that the formula holds for some $k \ge 1$, i.e.,

$$[Y_n^k]_{ij} = \sum_{1 \le i_2, \dots, i_k \le n} Y_{ii_2} Y_{i_2 i_3} \cdots Y_{i_k j}.$$

We must show that it holds for $k + 1$. Note that:

$$\begin{aligned}
[Y_n^{k+1}]_{ij} &= \sum_{\ell=1}^{n} [Y_n^k]_{i\ell} Y_{\ell j} \\
&= \sum_{\ell=1}^{n} \left(\sum_{1 \le i_2, \dots, i_k \le n} Y_{ii_2} Y_{i_2 i_3} \cdots Y_{i_k \ell} \right) Y_{\ell j} \\
&= \sum_{1 \le i_2, \dots, i_k, i_{k+1} \le n} Y_{ii_2} Y_{i_2 i_3} \cdots Y_{i_k i_{k+1}} Y_{i_{k+1} j}.
\end{aligned}$$

Thus, the formula holds for $k + 1$.

By induction, the result holds for all $k \ge 1$. □

Lemma 3.1.2 (Matrix Powers Trace)**.** *Let Y be an $n \times n$ matrix and $k \in \mathbb{N}$. Then, the trace of Y^k is given by:*

$$\mathrm{Tr}[Y^k] = \sum_{1 \le i_1, i_2, \dots, i_k \le n} Y_{i_1 i_2} Y_{i_2 i_3} \cdots Y_{i_k i_1}.$$

Proof. We can use the result from Lemma 3.1.1 to compute the trace of Y^k:

$$\mathrm{Tr}[Y^k] = \sum_{i=1}^{n}(Y^k)_{ii} = \sum_{i=1}^{n}(\sum_{1\le i_2,\dots i_k\le n} Y_{ii_2}Y_{i_2i_3}\dots Y_{i_ki})$$
$$= \sum_{1\le i_1,i_2,\dots,i_k\le n} Y_{i_1i_2}Y_{i_2i_3}\dots Y_{i_ki_1}.$$

□

Definition 3.1.3 (Graphs from Multi Index)**.** Let $\mathbf{i} \in [n]^k$ be a k-index, $\mathbf{i} = (i_1, i_2, \dots, i_k)$. A graph $G_\mathbf{i}$ is defined as follows: the vertices $V_\mathbf{i}$ are the distinct elements of

$$\{i_1, i_2, \dots, i_k\},$$

and the edges $E_\mathbf{i}$ are the distinct pairs among

$$\{i_1, i_2\}, \{i_2, i_3\}, \dots, \{i_{k-1}, i_k\}, \{i_k, i_1\}.$$

Definition 3.1.4 (Definition 4.2 in [1])**.** Let $\mathbf{i} \in [n]^k$ be a k-index, $\mathbf{i} = (i_1, i_2, \dots, i_k)$. the path $w_\mathbf{i}$ is the sequence

$$w_\mathbf{i} = (\{i_1, i_2\}, \{i_2, i_3\}, \dots, \{i_{k-1}, i_k\}, \{i_k, i_1\})$$

of edges from $E_\mathbf{i}$

Definition 3.1.5 (Graph Edge Count)**.** Let $\mathbf{i} \in [n]^k$ be a k-index, $\mathbf{i} = (i_1, i_2, \dots, i_k)$. For any edge $e = \{i, j\}$ from $E_\mathbf{i}$, we define the edge count $w_\mathbf{i}(e)$ as the number of times each edge e is traversed, and if $(i, j) \notin E_\mathbf{i}$, then $w_\mathbf{i}(\{i, j\}) = 0$.

Definition 3.1.6 (Matrix Multi Index)**.** Let $\mathbf{i} \in [n]^k$ be a k-index, $\mathbf{i} = (i_1, i_2, \dots, i_k)$. Let Y be a symmetric matrix. The matrix multi index $Y_\mathbf{i}$ is defined as:

$$Y_\mathbf{i} = Y_{i_1i_2}Y_{i_2i_3}\dots Y_{i_ki_1}$$

Lemma 3.1.7 (Matrix Multi Index and Graph Equivalence)**.** *Let $\mathbf{i} \in [n]^k$ be a k-index, $\mathbf{i} = (i_1, i_2, \dots, i_k)$. Let Y be a symmetric matrix. Then, we have the following:*

$$Y_\mathbf{i} = Y_{i_1i_2}Y_{i_2i_3}\dots Y_{i_ki_1} = \prod_{w\in w_\mathbf{i}} Y_w$$

Proof. We see from definition 3.1.4 that the path is defined as:

$$w_\mathbf{i} = (\{i_1, i_2\}, \{i_2, i_3\}, \dots, \{i_{k-1}, i_k\}, \{i_k, i_1\}).$$

if we use each each edge $w \in w_\mathbf{i}$, due to symmetry, we can write the product:

$$\prod_{w\in w_\mathbf{i}} Y_w = Y_{\{i_1,i_2\}}Y_{\{i_2,i_3\}}\dots Y_{\{i_{k-1},i_k\}}Y_{\{i_k,i_1\}} = Y_{i_1i_2}Y_{i_2i_3}\dots Y_{i_ki_1} = Y_\mathbf{i}$$

as required. □

Lemma 3.1.8 (Graph Walk and Graph Count Equivalence)**.** *Let $\mathbf{i} \in [n]^k$ be a k-index, $\mathbf{i} = (i_1, i_2, \dots, i_k)$. Let Y be a symmetric matrix. Then, we have the following:*

$$Y_\mathbf{i} = \prod_{w\in w_\mathbf{i}} Y_w = \prod_{1\le i\le j\le n} Y_{ij}^{w_\mathbf{i}(\{i,j\})}.$$

Proof. Using lemma 3.1.7, we already have that $Y_{\mathbf{i}} = \prod_{w \in w_{\mathbf{i}}} Y_w$. We consider the following cases for each (i, j):

$(i, j) \notin w_{\mathbf{i}}$: In this case, the entry $Y_{ij}^{w_{\mathbf{i}}(\{i,j\})} = 1$, making no contribution to the product.

$(i, j) \in w_{\mathbf{i}}$: In this case, the entry $Y_{ij}^{w_{\mathbf{i}}(\{i,j\})}$ is equal to the number of times the unordered edge $\{i, j\}$ appears in the path $w_{\mathbf{i}}$.

These two cover all possibilities, so putting them together we obtain

$$Y_{\mathbf{i}} = \prod_{1 \leq i \leq j \leq n} Y_{ij}^{w_{\mathbf{i}}(\{i,j\})}.$$

Indeed, if $(i, j) \notin w_{\mathbf{i}}$ then $w_{\mathbf{i}}(\{i, j\}) = 0$ and the corresponding factor contributes $Y_{ij}^0 = 1$, and if $(i, j) \in w_{\mathbf{i}}$ (hence also (j, i) if $j < i$), the exponent $w_{\mathbf{i}}(\{i, j\})$ counts exactly how many times the unordered edge $\{i, j\}$ appears in the walk, so the factor is $Y_{ij}^{w_{\mathbf{i}}(\{i,j\})}$.

Because every unordered pair $\{i, j\}$ with $1 \leq i \leq j \leq n$ is covered by one of these two cases and the product lists each edge exactly once. □

Definition 3.1.9 (Self Edges)**.** Let $\mathbf{i} \in [n]^k$ be a k-index. Given graph $G_{\mathbf{i}}$, define the self-edges $E_{\mathbf{i}}^s$ as:

$$\{\{i, i\} \in E_{\mathbf{i}}\},$$

the set of edges where both vertices are the same.

Definition 3.1.10 (Connecting Edges)**.** Let $\mathbf{i} \in [n]^k$ be a k-index. Given graph $G_{\mathbf{i}}$, define the connecting-edges $E_{\mathbf{i}}^c$ as:

$$\{\{i, j\} \in E_{\mathbf{i}} : i \neq j\}$$

Lemma 3.1.11 (Expectation of Matrix Multi Index)**.** *Let $\mathbf{i} \in [n]^k$ be a k-index, $\mathbf{i} = (i_1, i_2, \dots, i_k)$. Let Y be a symmetric matrix and let $\{Y_{ij}\}_{1 \leq i \leq j}$ be independent random variables, with $\{Y_{ii}\}_{i \geq 1}$ identically distributed and $\{Y_{ij}\}_{1 \leq i < j}$ identically distributed. Then, we have the following:*

$$\mathbb{E}(Y_{\mathbf{i}}) = \prod_{e_s \in E_{\mathbf{i}}^s} \mathbb{E}(Y_{11}^{w_{\mathbf{i}}(e_s)}) \cdot \prod_{e_c \in E_{\mathbf{i}}^c} \mathbb{E}(Y_{12}^{w_{\mathbf{i}}(e_c)}).$$

Proof. □

Lemma 3.1.12 (Trace of Expectation of Matrix)**.** *Let $\mathbf{Y}_n$ be an $n \times n$ symmetric matrix with independent entries, where $\{Y_{ij}\}_{1 \leq i \leq j}$ are independent random variables, with $\{Y_{ii}\}_{i \geq 1}$ identically distributed and $\{Y_{ij}\}_{1 \leq i < j}$ identically distributed. Then, for any k-index $\mathbf{i} \in [n]^k$, we have:*

$$\mathbb{E}(\mathrm{Tr}(\mathbf{Y}_n^k)) = \sum_{1 \leq i_1, i_2, \dots, i_k \leq n} \mathbb{E}(Y_{i_1 i_2} Y_{i_2 i_3} \cdots Y_{i_k i_1}) = \sum_{\mathbf{i} \in [n]^k} \mathbb{E}(Y_i)$$

Proof. Because each Y_{ij} is independent, we can write:

$$\mathbb{E}(Y_{\mathbf{i}}) = \mathbb{E}\left(\prod_{1 \leq i \leq j \leq n} Y_{ij}^{w_{\mathbf{i}}(\{i,j\})}\right) = \prod_{1 \leq i \leq j \leq n} \mathbb{E}(Y_{ij}^{w_{\mathbf{i}}(\{i,j\})})$$

using lemma 3.1.8. Consider the following cases for each (i, j) (we assume each $(i, j) \in w_{\mathbf{i}}$ since if they aren't, then $w_{\mathbf{i}}(i, j) = 0$ and this contributes nothing to the product):

$\mathbf{i} = \mathbf{j}$: in this case, we have $Y_{ij} = Y_{ii}$, and therefore the edge $(i,i) \in E^s_{\mathbf{i}}$. Since each Y_{ii} is identically distributed for all i, we have:

$$\mathbb{E}(Y_{ii}) = \mathbb{E}(Y_{11})$$

so the factor in the product above becomes $\mathbb{E}(Y_{ij}^{w_{\mathbf{i}}(\{i,j\})}) = \mathbb{E}(Y_{11}^{w_{\mathbf{i}}(i,i)})$.

$\mathbf{i} < \mathbf{j}$: in this case, we have $Y_{ij} \in E^c_{\mathbf{i}}$. Since each non diagonal entry Y_{ij} is identically distributed for all $i \neq j$ (and by symmetry $Y_{ij} = Y_{ji}$), we have:

$$\mathbb{E}(Y_{ij}) = \mathbb{E}(Y_{12})$$

so the factor in the product above becomes $\mathbb{E}(Y_{ij}^{w_{\mathbf{i}}(\{i,j\})}) = \mathbb{E}(Y_{12}^{w_{\mathbf{i}}(i,j)})$.

Plugging both of these cases into the earlier product over $E^c_{\mathbf{i}}$ and $E^s_{\mathbf{i}}$, we have:

$$\prod_{1 \le i \le j \le n} \mathbb{E}(Y_{ij}^{w_{\mathbf{i}}(\{i,j\})}) = \prod_{e_s \in E^s_{\mathbf{i}}} \mathbb{E}(Y_{11}^{w_{\mathbf{i}}(e_s)}) \cdot \prod_{e_c \in E^c_{\mathbf{i}}} \mathbb{E}(Y_{12}^{w_{\mathbf{i}}(e_c)})$$

as required.

□

Definition 3.1.13 (Product of Expectation of Matrix Multi Index)**.** Let $\mathbf{i} \in [n]^k$ be a k-index, $\mathbf{i} = (i_1, i_2, \ldots, i_k)$. Let Y be a symmetric matrix and let $\{Y_{ij}\}_{1 \le i \le j}$ be independent random variables, with $\{Y_{ii}\}_{i \ge 1}$ identically distributed and $\{Y_{ij}\}_{1 \le i < j}$ identically distributed. We define $\Pi(G_{\mathbf{i}}), w_{\mathbf{i}}$ as follows:

$$\Pi(G_{\mathbf{i}}, w_{\mathbf{i}}) = \prod_{e_s \in E^s_{\mathbf{i}}} \mathbb{E}(Y_{11}^{w_{\mathbf{i}}(e_s)}) \cdot \prod_{e_c \in E^c_{\mathbf{i}}} \mathbb{E}(Y_{12}^{w_{\mathbf{i}}(e_c)}) = \mathbb{E}(Y_{\mathbf{i}}).$$

Definition 3.1.14 (Length $|w_{\mathbf{i}}|$: R-1-1 : def:length_of_w_i)**.** Given a path $w_{\mathbf{i}}$ generated by some k-index $\mathbf{i}$, we let $|w_{\mathbf{i}}|$ denote the length of $w_{\mathbf{i}}$.

Lemma 3.1.15 ($|w_{\mathbf{i}}| = k$: R-1-2 : lem:abs_w_i_eq_k)**.** *For any k-index $\mathbf{i}$, the connected graph $G_{\mathbf{i}} = (V_{\mathbf{i}}, E_{\mathbf{i}})$ has at most k vertices. Furthermore*

$$|w_{\mathbf{i}}| \equiv \sum_{e \in E_{\mathbf{i}}} w_{\mathbf{i}}(e) = k.$$

Proof. Foremost, since the number of vertices of the graph $G_{\mathbf{i}}$ are the number of distinct elements of the k-index $\mathbf{i}$, it clearly follows that $\#V_{\mathbf{i}} \le k$. On the other hand, recall that each $w_i(e)$ denotes the number of times the edge $e \in E_{\mathbf{i}}$ is traversed by the path $w_{\mathbf{i}}$. Since $|w_{\mathbf{i}}| = k$ by the construction of $w_{\mathbf{i}}$, it follows that

$$|w_{\mathbf{i}}| \equiv \sum_{e \in E_{\mathbf{i}}} w_{\mathbf{i}}(e) = k.$$

□

Definition 3.1.16 (Length $|w|$: R-1-3 : def:length_of_w)**.** Given any graph $G = (V, E)$ and a path w, we let $|w|$ denote the length of w.

Definition 3.1.17 ($\mathcal{G}_k$: R-1-4 : def:g_k)**.** Let $\mathcal{G}_k$ denote the set of all ordered pairs (G, w) where $G = (V, E)$ is a connected graph with at most k vertices, and w is a closed path covering G satisfying $|w| = k$.

Lemma 3.1.18 (w can be uniquely expressed : R-1-5-0 : def:w_unique)**.** *Given* $(G, w) \in \mathcal{G}_k$*, let us denote the path* w *by*

$$w = (\{i_1, i_2\}, \{i_3, i_4\}, ..., \{i_{2k-3}, i_{2k-2}\}, \{i_{2k-1}, i_{2k}\}).$$

Then we can choose the smallest integer appearing in $\{i_1, i_2\} \cap \{i_{2k-1}, i_{2k}\}$ *such that* w *takes the form*

$$w = (\{j_1, j_2\}, \{j_2, j_3\}, ..., \{j_{k-1}, j_k\}, \{j_k, j_1\}). \tag{3.1}$$

Proof. There is at least one way and at most two ways to express w in the form of Equation 3.1. This follows from the fact that detemining the first entry j_1 (for which we have two choices of i_1 or i_2) of a path completely determines the remaining entries $j_2, j_3, ..., j_k$. Note that this also implies that we can not express express w in two 'distinct' forms of Equation 3.1 by starting with the same choice of j_1. □

Definition 3.1.19 (k-index generated by (G, w) : R-1-5 : def:g_k_j)**.** Given $(G, w) \in \mathcal{G}_k$, we denote $\mathbf{j}$ as the k-index generated by (G, w) in the following way. The path w can be uniquely expressed under the condition of Lemma 3.1.18:

$$w = (\{j_1, j_2\}, \{j_2, j_3\}, ..., \{j_{k-1}, j_k\}, \{j_k, j_1\}).$$

We define $\mathbf{j} = (j_1, j_2, ..., j_{k-1}, j_k)$.

Definition 3.1.20 ($(G_{\mathbf{i}}, w_{\mathbf{i}}) = (G, w)$: R-1-6 : def:g_k_equiv)**.** Let $(G_{\mathbf{i}}, w_{\mathbf{i}})$ be an ordered pair generated by some k-index $\mathbf{i}$ and $(G, w) \in \mathcal{G}_k$. We say $(G_{\mathbf{i}}, w_{\mathbf{i}}) = (G, w)$ if and only if there exists a bijection φ from the set of entries $\mathbf{i}$ onto the set of entries $\mathbf{j}$ such that

$$\mathbf{i} = (i_1, ..., i_k) \iff \mathbf{j} = (\varphi(i_1), \varphi(i_2), ..., \varphi(i_k)),$$

where $\mathbf{j}$ is a k-index generated by (G, w).

Lemma 3.1.21 ($\mathbf{i} \sim \mathbf{j} \Rightarrow \mathbb{E}(Y_{\mathbf{i}}) = \mathbb{E}(Y_{\mathbf{j}})$: R-1-7 : lem:eq_equiv_eq_expect)**.** *Given two* k*-indexes* $\mathbf{i} = (i_1, ..., i_k)$ *and* $\mathbf{j} = (j_1, ..., j_k)$*, suppose there exists a bijection* φ *from the set of entries of* $\mathbf{i}$ *onto the set of entries of* $\mathbf{j}$ *such that*

$$\mathbf{i} = (i_1, ..., i_k) \iff \mathbf{j} = (\varphi(i_1), \varphi(i_2), ..., \varphi(i_k)).$$

Then $\mathbb{E}(Y_{\mathbf{i}}) = \mathbb{E}(Y_{\mathbf{j}})$.

Proof. Given $Y_{\mathbf{i}} = Y_{i_1 i_2} Y_{i_2 i_3} \cdots Y_{i_{k-1} i_k} Y_{i_k i_1}$, we have

$$Y_{\mathbf{j}} = Y_{j_1 j_2} Y_{j_2 j_3} \cdots Y_{j_{k-1} j_k} Y_{j_k j_1} = Y_{\varphi(i_1)\varphi(i_2)} Y_{\varphi(i_2)\varphi(i_3)} \cdots Y_{\varphi(i_{k-1})\varphi(i_k)} Y_{\varphi(i_k)\varphi(i_1)}.$$

Observe that $\{i_{\lambda_l}, i_{\lambda_{l+1}}\}$ is a singleton and only if $\{\varphi(i_{\lambda_l}), \varphi(i_{\lambda_{l+1}})\}$ is a singleton. The fact that $\{i_{\lambda_l}, i_{\lambda_{l+1}}\} = \{i_{\lambda_\mu}, i_{\lambda_{\mu+1}}\}$ if and only if $\{\varphi(i_{\lambda_l}), \varphi(i_{\lambda_{l+1}})\} = \{\varphi(i_{\lambda_\mu}), \varphi(i_{\lambda_{\mu+1}})\}$ completes the proof. □

Definition 3.1.22 ($|G|$: R-1-8 : def:abs.G)**.** Given an ordered pair $(G, w) \in \mathcal{G}_k$, we define $|G|$ to be the number of distinct vertices in the graph G.

Lemma 3.1.23 (Lemma 4.3 in [1] : R-1-9 : lem:lem_4.3)**.** *Given* $(G, w) \in \mathcal{G}_k$*, we have*

$$\#\{\mathbf{i} \in [n]^k : (G_{\mathbf{i}}, w_{\mathbf{i}}) = (G, w)\} = n(n-1)\cdots(n-|G|+1).$$

Proof. By the way the equivalence relation is defined in Definition 3.1.17, the fact that there are $n(n-1)\cdots(n-|G|+1)$ ways to assign $|G|$ distinct values from $[n]$ into the indices $i_1, \dots, i_{|G|}$ completes the proof. ☐

Lemma 3.1.24 (Partitioning into double summation : R-1-10 : lem:equation_4.5_1)**.**

$$\mathbb{E}\,\mathrm{Tr}(\mathbf{Y}_{\mathbf{i}}^k) = \sum_{(G,w)\in\mathcal{G}_k} \sum_{\substack{\mathbf{i}\in[n]^k \\ (G_{\mathbf{i}}, w_{\mathbf{i}})=(G,w)}} \mathbb{E}(Y_{\mathbf{i}}).$$

Proof. This follows from 'partitioning' the summation appearing in Lemma 3.1.12 using the equivalence relation defined in Definition 3.1.20. ☐

Definition 3.1.25 ($\Pi(G, w)$: R-1-11 : def:Pi.G.w)**.** Given an ordered pair $(G, w) \in \mathcal{G}_k$, let $\mathbf{j}$ be the k-index generated by (G, w). We define

$$\Pi(G, w) = \mathbb{E}(Y_{\mathbf{j}}).$$

Lemma 3.1.26 (Re-indexing the sum with counting argument : R-1-12 : lem:equation_4.5_2)**.**

$$\mathbb{E}\,\mathrm{Tr}(\mathbf{Y}_{\mathbf{i}}^k) = \sum_{(G,w)\in\mathcal{G}_k} \Pi(G, w) \cdot \#\{\mathbf{i} \in [n]^k : (G_{\mathbf{i}}, w_{\mathbf{i}}) = (G, w)\}.$$

Proof. This follows from re-indexing the sum of Lemma 3.1.24 by using Lemma 3.1.21 and Lemma 3.1.23. ☐

Lemma 3.1.27 (Re-introducing the renormalization factor : R-1-13 : lem:equation_4.5_3)**.**

$$\frac{1}{n}\mathbb{E}\,\mathrm{Tr}(\mathbf{X}_n^k) = \sum_{(G,w)\in\mathcal{G}_k} \Pi(G, w) \cdot \frac{n(n-1)\cdots(n-|G|+1)}{n^{k/2+1}}.$$

Proof. Combining with the renormalization factor n^{-1} of Proposition **??** gives

$$\frac{1}{n}\mathbb{E}\,\mathrm{Tr}(\mathbf{X}_n^k) = \frac{1}{n^{k/2+1}}\mathbb{E}\,\mathrm{Tr}(\mathbf{Y}_{\mathbf{i}}^k).$$

Substituting the term $\mathbb{E}\,\mathrm{Tr}(\mathbf{Y}_{\mathbf{i}}^k)$ with the expression in Equation 3.1.26 gives

$$\frac{1}{n}\mathbb{E}\,\mathrm{Tr}(\mathbf{X}_n^k) = \sum_{(G,w)\in\mathcal{G}_k} \Pi(G, w) \cdot \frac{n(n-1)\cdots(n-|G|+1)}{n^{k/2+1}}.$$

☐

Definition 3.1.28 ($\mathcal{G}_{k,w\geq 2}$: R-1-14 : def:g_k_ge_2)**.** Let $\mathcal{G}_{k,w\geq 2}$ be a subset of $\mathcal{G}_k$ in which the walk w traverses each edge at least twice.

Lemma 3.1.29 ($\Pi(G, w) = 0$: R-1-15 : lem:Pi.prod_eq_zero_if_w_le_two)**.** *Given an ordered pair* $(G, w) \in \mathcal{G}_k$*, suppose there exists an edge* $e \in E$ *in which it is traversed only once in the walk* w*. Then*

$$\Pi(G, w) = 0.$$

Proof. Let $(G, w) \in \mathcal{G}_k$ and let $\mathbf{j}$ be the k-index generated by (G, w). Suppose there exists an edge $e \in E_{\mathbf{j}}$ such that $w_{\mathbf{j}}(e) = 1$. This means, in Lemma **??**, a singleton term $\mathbb{E}(Y_{ij}^{w_{\mathbf{j}}(e)}) = \mathbb{E}(Y_{ij})$ appears. The rest of the proof follows from the assumption of Proposition **??** that $\mathbb{E}(Y_{ij}) = 0$ for every i and j. □

Lemma 3.1.30 (Simplifying the summation with the fact $\Pi(G, w) = 0$ in certain cases: R-1-16 : lem:equation_4.8)**.**

$$\frac{1}{n}\mathbb{E}\operatorname{Tr}(\mathbf{X}_n^k) = \sum_{(G,w)\in\mathcal{G}_{k,w\geq 2}} \Pi(G, w) \cdot \frac{n(n-1)\cdots(n-|G|+1)}{n^{k/2+1}}.$$

Proof. This follows from applying the result of Lemma 3.1.29 to Lemma 3.1.27. □

Lemma 3.1.31 ($\#E \leq k/2$: R-1-17 : lem:edge_set_order_leq_k_over_two)**.** *Given an ordered pair $(G, w) \in \mathcal{G}_{k,w\geq 2}$, we must have $\#E \leq k/2$.*

Proof. Since $|w| = k$, if each edge in G is traversed at least twice, then by construction of w the number of edges is at most $k/2$. □

Proposition 3.1.32. *Let $G = (V, E)$ be a connected finite graph. Then, $|G| = \#V \leq \#E + 1$.*

Proof. $|G| = \#V \leq \#E + 1$: proof by induction on $\#V$. Base case $\#V = 1$ is obvious. For each additional vertex, the number of edges must increase by at least one for the graph to remain connected. □

Proposition 3.1.33. *Let $G = (V, E)$ be a connected finite graph. Then, $|G| = \#V = \#E + 1$ if and only if G is a plane tree.*

Proof. $|G| = \#V = \#E+1$ if G is a plane tree is already in Lean: SimpleGraph.IsTree.card_edgeFinset.

G is a plane tree if $|G| = \#V = \#E + 1$: proof by induction on $\#V$. Base case $\#V = 1$ has no edges. Assume $\#V = \#E + 1$ for $\#V = k$. Now, consider a tree with $\#V = k + 1$ nodes. Removing a leaf node leaves us with a tree with $\#V = k$ nodes. By IH, there are $k - 1$ edges. So including the leaf node gives us k edges. □

Lemma 3.1.34. *For any graph $G = (V, E)$ appearing in the sum in Equation 4.8, $|G| \leq k/2+1$.*

Proof. Follows directly from earlier lemmas (replacing $\#E$ with $k/2$). □

Lemma 3.1.35. $n(n-1)\cdots(n-|G|+1) \leq n^{|G|}$.

Proof. Use Nat.ascFactorial_eq_div. Or prove directly. □

Lemma 3.1.36. *The sequence $n \mapsto \frac{1}{n}\mathbb{E}\operatorname{Tr}(X_n^k)$ is bounded.*

Proof. The only part that depends on n is the big fraction. Since we only care about $w \geq 2$, $|G| \leq k/2 + 1$. Use the fact that the product $n(n-1)\cdots(n-|G|+1)$ is asymptotically equal to $n^{|G|}$ to conclude that the fraction is bounded and doesn't explode. □

Lemma 3.1.37. *Suppose k odd. Then, $|G| \leq \frac{k}{2} + \frac{1}{2}$.*

Proof. □

Lemma 3.1.38. *Suppose k odd. Then, $\frac{n(n-1)\cdots(n-|G|+1)}{n^{k/2+1}} \leq \frac{1}{\sqrt{n}}$.*

Proof. □

Proposition 3.1.39. *Suppose k odd. Then,* $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\operatorname{Tr}(X_n^k) = 0$.

Proof. Since $|G| \leq \#E + 1 \leq k/2 + 1$ and $|G|$ is an integer, it follows that $|G| \leq (k-1)/2 + 1 = k/2 + 1/2$. Hence, in this case, all the terms in the (finite n-independent) sum in Equation 4.8 are $O(n^{-1/2})$ □

Lemma 3.1.40. *If k is even and there exists e such that $w(e) \geq 3$, then $\#E \leq \frac{k-1}{2}$.*

Proof. □

Proposition 3.1.41. *Let $(G, w) \in \mathcal{G}_k$ with $w \geq 2$, and suppose k is even. If there exists a self-edge $e \in E_s$ in G, then $|G| \leq k/2$.*

Proof. Since the graph $G = (V, E)$ contains a loop, it is not a tree; it follows from Exercise **??** that $\#V < \#E + 1$. But $w \geq 2$ implies that $\#E \leq k/2$, and so $\#V < k/2 + 1$, and so $|G| = \#V \leq k/2$. □

Proposition 3.1.42. *Let $(G, w) \in \mathcal{G}_k$ with $w \geq 2$, and suppose k is even. If there exists an edge e in G with $w(e) \geq 3$, then $|G| \leq k/2$.*

Proof. The sum of w over all edges E in G is k. Hence, the sum of w over $E \setminus \{e\}$ is $\leq k-3$. Since $w \geq 2$, this means that the number of edges excepting e is $\leq (k-3)/2$; hence, $\#E \leq (k-3)/2+1 = (k-1)/2$. By the result of a previous lemma, this means that $\#V \leq (k-1)/2 + 1 = (k+1)/2$. Since k is even, it follows that $|G| = \#V \leq k/2$. □

Definition 3.1.43. Let $\mathcal{G}_k^{k/2+1}$ to be the set of pairs $(G, w) \in \mathcal{G}_k$ where G has $k/2 + 1$ vertices, contains no self-edges, and the walk w crosses every edge exactly 2 times.

Lemma 3.1.44. $|G_k|$ *is finite.*

Proof. Follows from definition. □

Lemma 3.1.45. *Elements of $G_k^{k/2+1}$ are trees.*

Proof. □

Lemma 3.1.46. *Elements of $G_k^{k/2+1}$ have $|E| = k/2$.*

Proof. □

Proposition 3.1.47. $|\sum_{\mathcal{G}_k, w\geq 2} - \sum_{\mathcal{G}_k^{k/2+1}}| \leq |\mathcal{G}_k|/n$.

Proof. □

Proposition 3.1.48. $\frac{1}{n}\mathbb{E}\operatorname{Tr}(X_n^k) = \sum_{(G,w)\in\mathcal{G}_k^{k/2+1}} \Pi(G, w) \cdot \frac{n(n-1)\cdots(n-|G|+1)}{n^{k/2+1}} + O_k(n^{-1})$

Proof. If $|G| < k/2+1$, then there is at least one more n in the denominator than the numerator. □

Proposition 3.1.49. $\lim_{n\to\infty} \frac{n^{k/2}}{n(n-1)\cdots(n-k/2+1)} = 1$.

Proof. some lower bound stuff + other stuff? □

Proposition 3.1.50. $\lim_{n\to\infty} \mathbb{E}\operatorname{Tr}(X_n^k) = \sum_{(G,w)\in\mathcal{G}_k^{k/2+1}} \Pi(G, w)$

Proof. Proof: use the fact that $|G| = k/2 + 1$ and $n(n-1)\cdots(n-k/2+1) \sim n^{k/2+1}$. Limit as n approaches infinity of $O(n^{-1/2})$ is 0. $\square$

Lemma 3.1.51. $\Pi(G, w) = \prod_{e_c \in E^c} \mathbb{E}(Y_{12}^{w(e_c)})$

Proof. Follows directly. $\square$

Lemma 3.1.52. $\prod_{e_c \in E^c} \mathbb{E}(Y_{12}^{w(e_c)}) = \prod_{e_c \in E^c} \mathbb{E}(Y_{12}^2)$

Proof. Follows directly. $\square$

Lemma 3.1.53. $\mathbb{E}(Y_{12}^2) = t^{\#E}$

Proof. Follows directly. $\square$

Lemma 3.1.54. $t^{\#E} = t^{k/2}$

Proof. Follows directly. $\square$

Proposition 3.1.55. $\Pi(G, w) = t^{k/2}$.

Proof. Proof slightly outdated.

Let $(G, w) \in \mathcal{G}_k^{k/2+1}$. Since w traverses each edge exactly twice, the number of edges in G is $k/2$. Since the number of vertices is $k/2 + 1$, Exercise (Prop) 4.3.1 shows that G is a tree. In particular there are no self-edges (as we saw already in Proposition 4.4).

1st equality: definition right after equation 4.4 in notes 2nd equality: proposition 4.4 (w < 3) and previous lemma (w = 1 –> 0) 3nd equality: definition (from main proposition) 4th equality: number of edges is k/2. $\square$

Proposition 3.1.56. $\lim_{n\to\infty} \mathbb{E}\,\mathrm{Tr}(X_n^k) = t^{k/2} \cdot \#\mathcal{G}_k^{k/2+1}$

Proof. Follows directly from 4.7.5 and 4.8. $\square$

Definition 3.1.57. A Dyck path of length k is a sequence $(d_1, ..., d_k) \in \{\pm 1\}^k$ such that their partial sum $\sum_{i=1}^{j} d_i \geq 0$ and total sum $\sum_{i=1}^{k} d_i = 0$. More intuitively, consider a diagonal lattice path from $(0, 0)$ to $(k, 0)$ consisting of $\frac{k}{2}$ ups and $\frac{k}{2}$ downs such that the path never goes below thw x-axis.

Definition 3.1.58. Define a map ϕ whose input is $(G, w) \in \mathcal{G}_k^{k/2+1}$. Then for its output, define a sequence $\mathbf{d} = \mathbf{d}(G, w) \in \{+1, -1\}^k$ recursively as follows. Let $d_1 = +1$. For $1 < j \leq k$, if $w_j \notin \{w_1, \dots, w_{j-1}\}$, set $d_j = +1$; otherwise, set $d_j = -1$; then $\mathbf{d}(G, w) = (d_1, \dots, d_k)$. $\phi((G, w)) = \mathbf{d}(G, w)$

Lemma 3.1.59. *$\phi((G, w)) = \mathbf{d}(G, w) \subseteq \mathcal{D}_k$, where $\mathcal{D}_k$ denotes the set of Dyck path of order k.*

Proof. set $P_0 = (0, 0)$ and $P_j = (j, d_1 + \cdots + d_j)$ for $1 \leq j \leq k$; then the piecewise linear path connecting $P_0, P_1, \dots, P_k$ is a lattice path. Since $(G, w) \in \mathcal{G}_k^2$, each edge appears exactly two times in w, meaning that the ± 1s come in pairs in $\mathbf{d}(G, w)$. Hence $d_1 + \cdots + d_k = 0$. What's more, for any edge e, the -1 assigned to its second appearance in w comes *after* the $+1$ corresponding to its first appearance; this means that the partial sums $d_1 + \cdots + d_j$ are all ≥ 0. That is: $\mathbf{d}(G, w)$ is a Dyck path $\square$

Definition 3.1.60. Define a map ψ whose input is a Dyck path of order k: $\mathbf{d} \in \{\pm 1\}^k$. Then the output is viewing this Dyck path as a contour reversal of a tree where an up $(d_i = 1)$ corresponds to visiting a child node and a down $(d_i = -1)$ corresponds to returning to parent node.

Lemma 3.1.61. $\psi(\mathbf{d}_k) \subseteq \mathcal{G}_k^{k/2+1}$

Proof. Use induction on the order of Dyck path k which is an even number. Assume $\phi(\mathbf{d}_{k-2}) \subseteq \mathcal{G}_{k-2}^{k/2-1}$. In the case of k, the last two steps appended to $\mathbf{d}_{k-2}$ has to be 1 followed by -1 in order for $\mathbf{d}_k$ to be a Dyck path. By induction hypothesis, this generates a graph with one extra vertex from the parent node, whose walk traversed at the last two steps of the walk. □

Lemma 3.1.62.

$$\phi \circ \psi = id_{\mathcal{D}_k}$$

Proof. Apply ψ to a given Dyck path $\mathbf{d}$ by definition, then apply ϕ to get a new sequence $\mathbf{d}'$ such that $d'_j = 1$ for new vertex, -1 otherwise. For the original Dyck path, the new vertex is the up step corresponding to 1. This implies ϕ recovers the original Dyck path. □

Lemma 3.1.63.

$$\psi \circ \phi = id_{\mathcal{G}_k^{k/2+1}}$$

Proof. The map ψ recovers the graph walk structure of the input from its Dyck path by the definition. □

Lemma 3.1.64. *Let k be even and let $\mathcal{D}_k$ denote the set of Dyck paths of length k*

$$\mathcal{D}_k = \{(d_1, \dots, d_k) \in \{\pm 1\} \colon \sum_{i=1}^{k} d_i \geq 0 \text{ for } 1 \leq j \leq j, \text{ and} \sum_{i=1}^{k} d_i = 0\}.$$

Then $(G, w) \mapsto d(G, w)$ is a bijection $\mathcal{G}_k^{k/2+1} \to \mathcal{D}_k$.

Proof. obvious from the previous lemmas. □

Definition 3.1.65.

$$C_0 = 1, \quad \text{and for } n \geq 1, \quad C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}.$$

Lemma 3.1.66. *#{binary trees with $\frac{k}{2}$ vertices} is given by Catalan number $C_{k/2}$*

Proof. □

Proposition 3.1.67.

$$|\mathcal{D}_k| = C_{k/2}$$

where $|\mathcal{D}_k|$ denotes the number of Dyke paths of length k while C_k is the kth Catalan number.

Proof. Given a binary tree with k nodes, perform preorder traversal: for each internal node visited, write an up-step $U = (1, 1)$. For each time return from a child, write a down-step $D = (1, -1)$. Since every internal node has exactly two children, there are $k/2$ U's and $k/2$ D's, giving a Dyke path of length k. Conversely, given a Dyck path, U is interpreted as adding new node while D is returning to the parent node. □

Proposition 3.1.68.

$$|\mathcal{G}_k^{k/2+1}| = C_{k/2}$$

Proposition 3.1.69 (Proposition 4.1 in [1])**.** *Let $\{Y_{ij}\}_{1\le i\le j}$ be independent random variables, with $\{Y_{ii}\}_{i\ge 1}$ identically distributed and $\{Y_{ij}\}_{1\le i<j}$ identically distributed. Suppose that $r_k = \max\{\mathbb{E}(|Y_{11}|^k), \mathbb{E}(|Y_{12}|^k)\} < \infty$ for each $k \in \mathbb{N}$. Suppose further than $\mathbb{E}(Y_{ij}) = 0$ for all i, j and set $t = \mathbb{E}(Y_{12}^2)$. If $i > j$, define $Y_{ij} \equiv Y_{ji}$, and let $\mathbf{Y}_n$ be the $n \times n$ matrix with $[\mathbf{Y}_n]_{ij} = Y_{ij}$ for $1 \le i, j \le n$. Let $\mathbf{X}_n = n^{-1/2}\mathbf{Y}_n$ be the corresponding Wigner matrix. Then*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\operatorname{Tr}(\mathbf{X}_n^k) = \begin{cases} t^{k/2}C_{k/2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}.$$

Proof. □

3.2 Convergence in Probability

Proposition 3.2.1 (Proposition 4.2 in [1])**.** *Let $\{Y_{ij}\}_{1\le i\le j}$ be independent random variables, with $\{Y_{ii}\}_{i\ge 1}$ identically distributed and $\{Y_{ij}\}_{1\le i<j}$ identically distributed. Suppose that $r_k = \max\{\mathbb{E}(|Y_{11}|^k), \mathbb{E}(|Y_{12}|^k)\} < \infty$ for each $k \in \mathbb{N}$. Suppose further than $\mathbb{E}(Y_{ij}) = 0$ for all i, j. If $i > j$, define $Y_{ij} \equiv Y_{ji}$, and let $\mathbf{Y}_n$ be the $n \times n$ matrix with $[\mathbf{Y}_n]_{ij} = Y_{ij}$ for $1 \le i, j \le n$. Let $\mathbf{X}_n = n^{-1/2}\mathbf{Y}_n$ be the corresponding Wigner matrix. Then*

$$\operatorname{Var}(\frac{1}{n}\operatorname{Tr}(\mathbf{X}_n^k)) = O_k(\frac{1}{n^2})$$

Chapter 4

SemicircleDistribution

4.1 Semicircle Probability Density Function

Definition 4.1.1. The function $f : \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(\mu, v, x) = \frac{1}{2\pi v}\sqrt{(4v - (x-\mu)^2)_+}$$

is called the probability density function (p.d.f.) of the semicircle distribution.

Lemma 4.1.2. *Given a mean $\mu \in \mathbb{R}$ and a variance $v \in \mathbb{R}_{\geq 0}$, the p.d.f. $f : \mathbb{R} \to \mathbb{R}$ of the semicircle distribution with mean μ and variance v is given by*

$$f(x) = \frac{1}{2\pi v}\sqrt{(4v - (x-\mu)^2)_+}.$$

Lemma 4.1.3. *If the variance v is given to be zero, then the p.d.f. of the semicircle distribution is the zero functional.*

Proof. By Definition 4.1.1, the square root of a nonpositive number is defined to be zero. Hence, the p.d.f. with a zero variance must be the zero functional. □

Lemma 4.1.4. *The p.d.f. of the semicircle distribution is always nonnegative.*

Proof. By Definition 4.1.1, the square root of a nonpositive number is defined to be zero. Furthermore, the variance is always assumed to be nonnegative. Therefore, since the fractional term and the square root term are always nonnegative, we conclude the p.d.f. is always nonnegative. □

Lemma 4.1.5. *Given a mean $\mu \in \mathbb{R}$ and a variance $v \in \mathbb{R}_{\geq 0}$, the p.d.f. $f : \mathbb{R} \to \mathbb{R}$ of the semicircle distribution with mean μ and variance v is measurable.*

Proof. □

Lemma 4.1.6. *Given a mean $\mu \in \mathbb{R}$ and a variance $v \in \mathbb{R}_{\geq 0}$, the p.d.f. $f : \mathbb{R} \to \mathbb{R}$ of the semicircle distribution with mean μ and variance v is strongly measurable.*

Proof. By Lemma 4.1.5, we know the p.d.f. f with fixed mean μ and variance v is measurable. Since $\mathbb{R}$ is equipped with a second countable topology, the fact that f with fixed mean μ and variance v implies f is strongly measurable. □

Lemma 4.1.7. *Given a mean $\mu \in \mathbb{R}$ and a variance $v \in \mathbb{R}_{\geq 0}$, the p.d.f. $f : \mathbb{R} \to \mathbb{R}$ of the semicircle distribution with mean μ and variance v is integrable.*

Proof. □

Lemma 4.1.8. *Given a mean $\mu \in \mathbb{R}$ and a nonzero variance $v \in \mathbb{R}_{>0}$, the lower integral of the p.d.f. $f : \mathbb{R} \to \mathbb{R}$ of the semicircle distribution with mean μ and variance v equals 1.*

Proof. □

Lemma 4.1.9. *Given a mean $\mu \in \mathbb{R}$ and a nonzero variance $v \in \mathbb{R}_{>0}$, the integral of the p.d.f. $f : \mathbb{R} \to \mathbb{R}$ of the semicircle distribution with mean μ and variance v equals 1.*

Proof. □

Lemma 4.1.10. *For any p.d.f. $f : \mathbb{R} \to \mathbb{R}$ of the semicircle distribution, the following relation is satisfied:*

$$f(\mu, v, x - y) = f(\mu + y, v, x)$$

for any $u \in \mathbb{R}$, $v \in \mathbb{R}_{\geq 0}$, and $x, y \in \mathbb{R}$.

Proof. Expanding Definition 4.1.1 gives

$$f(\mu, v, x - y) = \frac{1}{2\pi v}\sqrt{\left(4v - ((x - y) - \mu)^2\right)_+} = \frac{1}{2\pi v}\sqrt{\left(4v - (x - (\mu + y))^2\right)_+} = f(\mu + y, v, x).$$

□

Lemma 4.1.11. *For any p.d.f. $f : \mathbb{R} \to \mathbb{R}$ of the semicircle distribution, the following relation is satisfied:*

$$f(\mu, v, x + y) = f(\mu - y, v, x)$$

for any $u \in \mathbb{R}$, $v \in \mathbb{R}_{\geq 0}$, and $x, y \in \mathbb{R}$.

Proof. Expanding Definition 4.1.1 gives

$$f(\mu, v, x + y) = \frac{1}{2\pi v}\sqrt{\left(4v - ((x + y) - \mu)^2\right)_+} = \frac{1}{2\pi v}\sqrt{\left(4v - (x - (\mu - y))^2\right)_+} = f(\mu - y, v, x).$$

□

Lemma 4.1.12. *For any p.d.f. $f : \mathbb{R} \to \mathbb{R}$ of the semicircle distribution, the following relation is satisfied:*

$$f(\mu, v, c^{-1}x) = |c| \cdot f(c\mu, c^2 v, x)$$

for any $u \in \mathbb{R}$, $v \in \mathbb{R}_{\geq 0}$, $x \in \mathbb{R}$, and nonzero $c \in \mathbb{R}$.

Proof. □

Lemma 4.1.13. *For any p.d.f. $f : \mathbb{R} \to \mathbb{R}$ of the semicircle distribution, the following relation is satisfied:*

$$f(\mu, v, cx) = |c^{-1}| \cdot f(c^{-1}\mu, c^{-2}v, x)$$

for any $u \in \mathbb{R}$, $v \in \mathbb{R}_{\geq 0}$, $x \in \mathbb{R}$, and nonzero $c \in \mathbb{R}$.

Proof. Expanding Definition 4.1.1 gives

$$f(\mu, v, cx) = \frac{1}{2\pi v}\sqrt{\left(4v - (cx - \mu)^2\right)_+} = \frac{1}{2\pi v}\sqrt{\left(4v - c^2(x - c^{-1}\mu)^2\right)_+} = |c^{-1}|\frac{1}{2\pi(c^{-2}v)}\sqrt{\left(4(c^{-2}v) - (x - c^{-1}\mu)^2\right)_+} =$$

□

4.2 To Extended Nonnegative Reals

Definition 4.2.1. Let $f : \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ denote the real-valued semicircle density defined in Definition 4.1.1. Define the function $h : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ as follows:

$$h(x) := \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we define the function $g : \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R} \to [0, \infty] \subseteq \overline{\mathbb{R}}_{\geq 0}$ by:

$$g(\mu, v, x) := h(f(\mu, v, x)),$$

Lemma 4.2.2. *For all $\mu \in \mathbb{R}$, $v \in \mathbb{R}_{\geq 0}$, the extended p. d. f. $g : \mathbb{R} \to [0, \infty]$ satisfies:*

$$g(\mu, v) = (x \mapsto h(f(\mu, v, x)))\,.$$

Lemma 4.2.3. *If the variance v is zero, then the extended p.d.f. is identically zero:*

$$\forall x \in \mathbb{R}, \quad g(\mu, 0, x) = 0.$$

Proof. This follows immediately from the definition of g as $h(f(\mu, 0, x))$, and the fact that $f(\mu, 0, x) = 0$ from Lemma 4.1.3. □

Lemma 4.2.4. *Let $\mu \in \mathbb{R}$, $v \in \mathbb{R}_{\geq 0}$, and $x \in \mathbb{R}$. Then the real value recovered from the extended semicircle PDF satisfies:*

$$g(\mu, v, x)^{\text{toReal}} = f(\mu, v, x).$$

Proof. Since $f(\mu, v, x) \geq 0$, we have $h(f(\mu, v, x)) = f(\mu, v, x)$, and thus

$$g(\mu, v, x) = h(f(\mu, v, x)) = f(\mu, v, x).$$

Therefore,

$$g(\mu, v, x)^{\text{toReal}} = f(\mu, v, x),$$

as desired. □

Lemma 4.2.5. *If $v > 0$, then for all $\mu, x \in \mathbb{R}$, the extended p.d.f. is nonnegative:*

$$0 \leq g(\mu, v, x).$$

Proof. This is immediate from the definition of g as $h(f(\mu, v, x))$ and the nonnegativity of f (Lemma 4.1.4). □

Lemma 4.2.6. *For all $\mu, x \in \mathbb{R}$, and $v \in \mathbb{R}_{\geq 0}$, we have:*

$$g(\mu, v, x) < \infty.$$

Proof. Since $f(\mu, v, x) \in \mathbb{R}_{\geq 0}$, we have $g(\mu, v, x) = h(f(\mu, v, x)) < \infty$. □

Lemma 4.2.7. *For all $\mu, x \in \mathbb{R}$, and $v \in \mathbb{R}_{\geq 0}$, the extended p.d.f. is finite:*

$$g(\mu, v, x) \neq \infty.$$

Lemma 4.2.8. *Let $\mu \in \mathbb{R}$ and $v \in \mathbb{R}_{>0}$. Then the support of the extended p.d.f. is*

$$\operatorname{supp}(g(\mu, v)) = \{x \in \mathbb{R} : f(\mu, v, x) \neq 0\} = \left[\mu - 2\sqrt{v}, \mu + 2\sqrt{v}\right].$$

Proof. sorry ☐

Lemma 4.2.9. *The function $x \mapsto g(\mu, v, x)$ is measurable for all $\mu \in \mathbb{R}$, $v \in \mathbb{R}_{\geq 0}$.*

Proof. Since h is measurable, and h is a measurable map $\mathbb{R}_{\geq 0} \to \overline{\mathbb{R}}_{\geq 0}$, their composition is measurable. ☐

Lemma 4.2.10. *If $v > 0$, then the total integral of g with respect to Lebesgue measure is 1:*

$$\int_{\mathbb{R}} g(\mu, v, x)\, dx = 1.$$

Proof. This follows from the equality:

$$\int_{\mathbb{R}} h(f(\mu, v, x))\, dx = h(\left(\int_{\mathbb{R}} f(\mu, v, x)\, dx\right) = h(1) = 1$$

using Lemma 4.1.8. ☐

4.3 Semicircle Distribution

Definition 4.3.1. The semicircle distribution with mean μ and variance v is the Dirac delta at if v = 0; otherwise, it's the Lebesgue measure weighted by the semicircle PDF.

Lemma 4.3.2. *If $v \neq 0$, then the definition the semicircle distribution is defined as the Lebesgue measure weighted by the semicircle probability density function.*

Proof. Follows directly from definition of semicircle distribution. ☐

Lemma 4.3.3. *If the variance is 0, then the semicircle distribution is exactly the Dirac measure at μ.*

Proof. Follows directly from definition of semicircle distribution. ☐

Lemma 4.3.4. *The measure semicircleReal is a probability measure, no matter the values of $\mu \in \mathbb{R}$ and $v \in \mathbb{R}_{\geq 0}$.*

Lemma 4.3.5. *If the variance v is nonzero, then the semicircle distribution has no atoms.*

Lemma 4.3.6. *For a semicircle measure with mean μ and nonzero variance v, the measure of any measurable set s equals the Lebesgue integral over s of the semicircle probability density function at x.*

Lemma 4.3.7. *For any real mean μ, and any nonnegative variance v that is not zero, and any measurable set s of real numbers, the semicircle distribution measure of the set s equals the extended nonnegative real number version (ENNReal.ofReal) of the integral of the semicircle probability density function over s.*

Lemma 4.3.8. *For a semicircle distribution with mean μ and nonzero variance v, the measure semicircleReal μ v is absolutely continuous with respect to the Lebesgue measure.*

Lemma 4.3.9. *The Radon–Nikodym derivative of the semicircle measure semicircleReal μ v with respect to the Lebesgue measure is almost everywhere equal to the semicircle probability density function semicirclePDF (μ, v).*

Lemma 4.3.10.

Let $f : \mathbb{R} \to E$ be a function where E is a normed vector space over $\mathbb{R}$. For the semicircle distribution with mean $\mu \in \mathbb{R}$ and variance $v > 0$, we have:

$$\int f(x)\, d(\text{semicircleReal } \mu\, v)(x) = \int \text{semicirclePDFReal}(\mu, v, x) \cdot f(x)\, dx.$$

4.4 Transformations

Lemma 4.4.1. *The map of a semicircle distribution by addition of a constant is semicircular. That is, given a constant $y \in \mathbb{R}$, $\mathrm{SC}(\mu, v) \circ (X \mapsto X + y)^{-1} = \mathrm{SC}(\mu + y, v)$.*

Proof. □

Lemma 4.4.2. *The map of a semicircle distribution by addition of a constant is semicircular. That is, given a constant $y \in \mathbb{R}$, $\mathrm{SC}(\mu, v) \circ (X \mapsto y + X)^{-1} = \mathrm{SC}(y + \mu, v)$.*

Proof. Obvious from commutativity between $X + y$ and $y + X$. □

Lemma 4.4.3. *The map of a semicircle distribution by multiplication by a constant is semicircular. That is, given a constant $c \in \mathbb{R}$, $\mathrm{SC}(\mu, v) \circ (X \mapsto cX)^{-1} = \mathrm{SC}(c\mu, c^2 v)$.*

Proof. □

Lemma 4.4.4. *The map of a semicircle distribution by multiplication by a constant is semicircular. That is, given a constant $c \in \mathbb{R}$, $\mathrm{SC}(\mu, v) \circ (X \mapsto Xc)^{-1} = \mathrm{SC}(\mu c, c^2 v)$.*

Proof. Use commutativity between Xc and cX. □

Lemma 4.4.5. *Given a constant $c \in \mathbb{R}$, $\mathrm{SC}(\mu, v) \circ (X \mapsto -X)^{-1} = \mathrm{SC}(-\mu, v)$*

Proof. Special case of the multiplication by constant map with constant being -1. □

Lemma 4.4.6. *The map of a semicircle distribution by multiplication by a constant is semicircular. That is, given a constant $y \in \mathbb{R}$, $\mathrm{SC}(\mu, v) \circ (X \mapsto X - y)^{-1} = \mathrm{SC}(\mu - y, v)$*

Proof. Use the map by addition of constant and substitute constant for its -1 multiple. □

Lemma 4.4.7. *The map of a semicircle distribution by multiplication by a constant is semicircular. That is, given a constant $y \in \mathbb{R}$, $\mathrm{SC}(\mu, v) \circ (X \mapsto y - X)^{-1} = \mathrm{SC}(y - \mu, v)$*

Proof. □

Lemma 4.4.8. *Given a real random variable $X \sim \mathrm{SC}(\mu, v)$ then for a constant $y \in \mathbb{R}$, $X + y \sim \mathrm{SC}(\mu + y, v)$*

Proof. □

Lemma 4.4.9. *Given a real random variable $X \sim \mathrm{SC}(\mu, v)$ then for a constant $y \in \mathbb{R}$, $X + y \sim \mathrm{SC}(y + \mu, v)$*

Proof. □

Lemma 4.4.10. *Given a real random variable $X \sim \mathrm{SC}(\mu, v)$, then for a constant $c \in \mathbb{R}$, $cX \sim \mathrm{SC}(c\mu, c^2 v)$*

Proof. $\square$

Lemma 4.4.11. *Given a real random variable* $X \sim \mathrm{SC}(\mu, v)$*, then for a constant* $c \in \mathbb{R}$*,* $Xc \sim \mathrm{SC}(\mu c, c^2 v)$

Proof. $\square$

Lemma 4.4.12.

$$\mathbb{E}[X] = \int x d\sigma = \mu$$

Lemma 4.4.13. $Var(X) = v$

Lemma 4.4.14. *The variance of a real semicircle distribution with parameter* (μ, v) *is its variance parameter* v

Lemma 4.4.15. *All the moments of a real semicircle distribution are finite. That is, the identity is in* L_p *for all finite* p

Lemma 4.4.16. $\mathbb{E}[(X-\mu)^{2n}] = v_n C_n$

Lemma 4.4.17. $\mathbb{E}[(X-\mu)^{2n}] = v_n C_n$

Lemma 4.4.18. $\mathbb{E}[(X-\mu)^{2n+1}] = 0$

Lemma 4.4.19. $\mathbb{E}[(X-\mu)^{2n+1}] = 0$

Bibliography

[1] Todd Kemp. Math 247a: Introduction to random matrix theory. *Lecture notes*, pages 3–4, 2013.